A THEORY OF STATISTICAL DECISION UNDER UNCERTAINTY—THE 'BENEFIT' CRITERION

BY R.C. AGRAWAL[1] & E.O. HEADY[2]

(*Received in April, 1969*)

This paper deals with the problem of decision making by an individual under uncertainty. The individual can be a person, firm or a group of persons working as a unit with a single objective. Hence, the problem can be treated as a one-person game, with 'nature' as the other or 'passive' player. We first review four conventional theories of choice which can serve for decision making under uncertainty, we then suggest a new theory, *the benefit criterion*, which in general appears to have merit over conventional theories.

DECISION FRAMEWORK

A decision making problem under uncertainty has the following four basic components relating to the decision maker : (*a*) an objective function, (*b*) a set of strategies or alternative courses of action, (*c*) payoffs or outcomes associated with given strategies of the decision maker for each state of nature, and (*d*) uncertainty about the state of nature likely to prevail in the period for which the decision is made. The problem can be summarized as follows :

Let

$$S=(s_1, s_2,\ldots, s_i, \ldots s_m)$$

be the strategy set of the decision maker,

$$T=(t_1, t_2,\ldots, t_j,\ldots t_n)$$

be the states of nature, and $P=\{p_{ij}\}$ be the payoff matrix of the decision maker. Thus P lists each outcome, p_{ij}, associated with the ith strategy of the decision maker when the jth state of nature prevails.

---

1. Associate Professor in Agricultural Economics, U.P. Agricultural University, Pantnagar.

2. Curtiss Distinguished Professor, Iowa State University, Ames, Iowa.

Then given the objective function of the individual decision maker and P, the problem is reduced to the choice of suitable s, $s=\{s_i/s_i \in S\}$, such that 's' optimizes the objective function.

Under complete information about the future, no uncertainty is involved in decision making and the choice of a suitable 's' may be trivial. If knowledge of the future is not perfect, the problem becomes very complex since present decisions fall in the realm of uncertainty. Uncertainty is a subjective phenomenon because the parameters of relevant probability distributions cannot be established empirically. Therefore, given the same set of circumstances, two decision makers may not visualize future happenings in the same manner and the decision of A may differ from that of B. As he looks to the future, a decision maker forms expectations of the consequences of his decisions. Hence, he does not maximize his payoff or utility function but maximizes their expected value.

CONVENTIONAL CRITERIA

Theories of statistical decision deal with the 'best-looking' course of action under incomplete information due to uncertainty where the decision maker has no idea (no probability basis) of the state of nature likely to prevail. Four conventional criteria or theories of choice for selecting optimal strategies ('s') in the realm of uncertainty are : (*a*) Wald's maximin criterion, (*b*) Laplace's principle of insufficient reason, (*c*) Savage's Regret criterion, and (*d*) Hurwicz' 'optimism-pessimism' criterion.

To aid in evaluating the 'benefit' criterion, we briefly illustrate these four theories and indicate their advantages and limitations. All these theories suggest the strategy set s that maximizes the expected utility of the decision maker. It is the definition of 'expected utility' that varies from criterion to criterion. Accordingly, as we shall demonstrate, the four different criteria may specify different optimal strategies for a given payoff matrix.

Throughout this paper, the following assumptions have been made :

(*a*) The game is of the form $G=(S, T, P)$ where S, T and P are the same as on page 2.

(*b*) i and j are finite.

(*c*) The utilities can be numerically expressed. This is quite a restrictive assumption. However, we can relax this assumption for the 'benefit' criterion without serious difficulties.

(*d*) The alternative acts can be ranked according to the utilities associated with them.

(*e*) Transitivity of acts and the outcomes, *i.e.* if p_{i*j} is preferred to p_{ij*} and if p_{ij*} is preferred to p_{i*j*} then p_{i*j} is preferred to p_{i*j*}.

(*f*) The decision maker is an individual, a firm or a group of individuals working together as a single unit for the same objective.

(*g*) The decision maker maximizes the expected utility of his strategies.

Wald's Maximin Criterion

This is a conservative criterion where the decision maker attaches a probability of one to the worst consequence for a given s_i and zero to the other outcomes in that row. Let $\overset{W}{E}(u_i)$ be the expected utility of his ith strategy (s_i) to the decision maker under Wald's criterion. Then

$$\overset{W}{E}(u_i)=\min_j p_{ij}. \quad \text{If } \max_i \overset{W}{E}(u_i)=\overset{W}{E}(u_{i*}),$$

the i^*th strategy is optimal to the decision maker.

Laplace's Principle of Insufficient Reason

The theory assumes complete ignorance on the part of the decision maker about the state of nature that will prevail. Hence, it is assumed that each state of nature is equally probable. Let $\overset{L}{E}(u_i)$ be the expected utility of the ith strategy to the decision maker under the Laplace's principle. Then

$$\overset{L}{E}(u_i)=n^{-1}\sum_{j=1}^{n} p_{ij}.$$

Again, if $\max_i \overset{L}{E}(u_i)=\overset{L}{E}(u_i{}^*)$, the decision maker will choose the i^*th strategy, s_{i*}.

Hurwicz' 'Optimism-pessimism' Criterion

According to this criterion, the decision maker assigns a probability of a, $0\leqslant a\leqslant 1$, to the best outcome for a given s_i and a probability of '$1-a$' to the worst outcome in that row.

Let $\overset{H}{E}(u_i)$ be the expected utility to the decision maker of his ith strategy under Hurwicz' model. Then

$$\overset{H}{E}(u_i)=(a)\ (\max_j p_{ij})+(1-a)\ (\min_j p_{ij}).$$

If $\max \overset{H}{E}(u_i)=\overset{H}{E}(u_{i*})$, the choice will fall on s_{i*}.

SAVAGE'S 'REGRET' CRITERION

The behavioral assumption under this criterion is that the decision maker tries to minimize his risk or 'regret' where regret is defined as the difference between the actual payoff for the ith strategy and the maximum payoff that he could have obtained if he had an advance knowledge of the true state of nature that actually prevailed. Let R be the Regret matrix with elements r_{ij}. Then for a given state of nature t_{jo}, $r_{ijo}=p_{ijo}-\max_i p_{ijo}$. Clearly $r_{ij}\leqslant 0$.

Let $\overset{R}{E}(u_j)$ be the expected utility to the decision maker of his ith strategy as defined by the Regret criterion. Then

$$\overset{R}{E}(u_i)=\min_j r_{ij}.$$

If $\max_i \overset{R}{E}(u_i)=\overset{R}{E}(u_{i*})$, s_{i*} is optimal to the decision maker under Regret criterion.

ILLUSTRATION OF THE FOUR CRITERIA

We illustrate the mechanics of the four theories of choice for the following payoff matrix :

States of nature

		t_1	t_2	t_3
$P=$ Strategies of decision Maker	s_1	7	13	5
	s_2	10	9	6
	s_3	8	7	8.5

...(1)

If we assume $a=\cdot 3$, then $(1-a)=\cdot 7$. Therefore, $\overset{W}{E}(u_i)$ $\overset{L}{E}u_i)$ and $\overset{H}{E}(u_i)$ are as given below :

	t_1	t_2	t_3	$\overset{W}{E}(u_i)$	$\overset{L}{E}(u_i)$	$\overset{H}{E}(u_i)$ for $a=3$
s_1	7	13	5	5	$\frac{7+13+5}{3}=\frac{25^*}{3}$	$(\cdot 3)(13)+(1-\cdot 3)(5)=7\cdot 4$
s_2	10	9	6	6	$\frac{10+9+6}{3}=\frac{25^*}{3}$	$(\cdot 3)(10)+(1-\cdot 3)(6)=7\cdot 2$
s_3	8	7	8·5	7*	$\frac{8+7+8\cdot 5}{3}=\frac{23\cdot 5}{3}$	$(\cdot 3)(8\cdot 5)+(1-\cdot 3)(7)=7\cdot 45$

The 'Regret' matrix for the P in (1) and the respective $E(u_i^R)$ are given in (2)

$$R = \begin{array}{c} s_1 \\ s_1 \\ s_3 \end{array} \overset{\begin{array}{ccc} t_1 & t_2 & t_2 \end{array}}{\begin{bmatrix} -3 & 0 & -3{\cdot}5 \\ 0 & -4 & -2{\cdot}5 \\ -2 & -6 & 0 \end{bmatrix}} \quad \begin{array}{c} E(u_i^R) \\ -3{\cdot}5^* \\ -4 \\ -6 \end{array} \qquad \ldots(2)$$

To sum up, for the pay off matrix P_i the following are the optimal strategies under the different theories of choice :

Criterion	Optimal strategy/strategies
Wald's	s_3
Laplace's	s_1 and/or s_2
Hurwicz (for $a = {\cdot}3$)	s_3
Regret	s_1

Comparison of the four theories of choice

Wald's model chooses as the optimal that strategy which affords the maximum security level to the decision maker under the assumption that nature will try to do the worst to him. This is hardly true. Moreover, we are concerned only with a one-person game where the nature has been assumed to be 'passive'. The decision makes acts as a risk-averter under this criterion. Since the theory places all the weight on the worst outcome in a row and none on others, it may lead to ridiculous results. Consider the payoff matrix in (3).

$$\bar{p} = \begin{array}{c} s_1 \\ s_2 \\ s_3 \end{array} \overset{\begin{array}{cc} t_1 & t_2 \end{array}}{\begin{bmatrix} 10 & 15 \\ 8 & 300 \\ 7 & 1000 \end{bmatrix}} \qquad \ldots(3)$$

Wald's criterion would always suggest s_1, no matter whether we have 30, 300, 3000 or 3 million as p_{22}. For p in (3), as long as the probability of t_2 is more than 2/287, s_2 should be preferred to s_1. Similarly, unless the probability of t_1 is as high as 985/988, a prudent decision maker should prefer s_3 over s_1.

Laplace's principle is suitable where data are available for all pertinent states of 'nature'. Rather than use the Laplace principle in such a case, however, one could do better by applying usual statistical procedures and test of significance to specify the best strategy. Of the four theories, Laplace's criterion is more appropriate for long-run decisions because the longer the period, the "better the operation of law of averages". However, in practice not enough data may be available to establish the probabilities of the states of nature. It is under such circumstances that a theory of statistical decision is needed.

Hurwicz's model takes only the extreme values in a row into consideration and completely ignores the rest of the data. In this respect, it is similar to 'range' as a measure of dispersion. For P in (1), p_{11} could take any value from 5 to 13, p_{22} could be anything between 6 and 10 and p_{31} could assume any value between 7 and 8·5 and still s_3 would be optimal under the Hurwicz model. A serious difficulty may arise in choosing an appropriate 'a' which is greatly affected hy individual's judgement, psychology, outlook and education. Also, it is doubtful that the same 'a' should be specified for all s_i, the model thus becoming highly subjective. At $a=0$, the Hurwicz approach is identical to that of Wald.

The 'Regret' criterion assumes that the decision maker is risk-conscious *i.e.* tries to minimize his risk. It may, though not necessarily, provide the decision maker with the highest payoff in P. For P in (1), the Regret criterion specifics s_1 as optimal. If the decision maker employs s_1 aud if the state of nature is t_2, the payoff to the decision maker would be as high as 13. However, there is also a danger of his getting only 5, the minimum in p, if t_3 comes to prevail. The minimum payoff under Wald's approach is at least as high as that under any other criterion. But the decision maker may never be able to achieve the maximum payoff by using Wald's model. Thus, the Wald and Regret criteria represent two extremes. For this reason, we suggest a new theory of choice—the criterion of 'benefit' —which blends the properties of these two models.

The Criterion of 'Benefit'

In addition to the assumptions made earlier, the following behaviour is assumed on the part of the decision maker.

For a given state of nature, t_{jo}, he determines his own strategy, s_{io}, which has the smallest payoff under t_{jo}. If t_{jo} prevails, the worst realization occurs under the use of his ioth strategy, s_{io}. If he chooses a strategy s_i, $i \neq io$, he certainly will gain over the worst, s_{io}, under t_{jo}. To what extent is he better off? Under assumption (c), on page 72, we suppose that he finds the amount by deducting the lowest payoff for t_{j0} (*i.e.* p_{i0j0}) from the payoff under t_{j0} if he employs s_i. This difference is termed as the 'benefit' or additional return resulting from his wisdom in choosing a strategy other than the worst for the given state of nature.

Let B be the 'benefit' matrix of the decision maker with elements b_{ij} where $b_{ij0}=p_{ij0}-\min_i p_{ij0}$. Clearly, $b_{ij} \geqslant 0$. Further, the dimensions of B and P are the same.

If $E(u_i^B)$ is the expected utility to the decision maker of his ith strategy under the 'benefit' criterion, then we define $E(u_i^B)=\min b_{ij}$. Again if $\max_i E(u_i^B) = E(u_{i*}^B)$, then s_{i*} is the optimal strategy under the 'benefit' theory. For p in (1) the 'benefit' matrix is calculated in the following manner :

(*i*) for t_1 (*i.e.* $j^0=1$), $\min_i p_{i1}=p_{11}=7$ and $i^0=1$.

Therefore,

$$b_{11}=p_{11}-p_{11}= 7-7=0$$
$$b_{21}=p_{21}-p_{11}=10-7=3$$
$$b_{31}=p_{31}-p_{11}= 8-7=1$$

Similarly,

(*ii*) for t_2, $\min_i p_{12}=p_{32}=7$ and, therefore,

$$b_{12}=6,\ b_{22}=2 \text{ and } b_{32}=0, \text{ and}$$

(*iii*) for t_2, $\min_i p_{13}=p_{13}=5$ and,

$$b_{13}=0,\ b_{23}=1 \text{ and } b_{33}=3{\cdot}5.$$

The B matrix and the respective $E(u_1^B)$ are given in (4).

$$B=\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \end{array}\begin{array}{c} \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ \begin{bmatrix} 0 & 6 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 3{\cdot}5 \end{bmatrix} \end{array} \quad \begin{array}{c} E(u_1B) \\ 0 \\ 1^* \\ 0 \end{array} \qquad \ldots(4)$$

$\max E(u_i^B)=E(u_{i*}^B)=E(u_2^B)$ and, according to the 'benefit' criterion, s_2 is the optimal strategy.

In case of a tie among the $E(u_{i*}^B)$, the tie may be broken by trying the next higher (than the minimum) payoff for the rows in the tie and then choosing the one with the higher figure. For example, the minimum for both s_1 and s_3 is zero in B. If we make a choice between these two, we should look for the value higher than zero. It is 6 for s_1 and 1 for s_3 and with $6>1$, we choose s_1. As noted earlier, Wald's model chose s_3 whereas s_1 was optimal under the Regret approach. Thus, in our example, each of the three principles of choice suggests a different strategy as optimal.

Reasons for different choices

Two questions suggest themselves : (*a*) Why do these three criterion give different results in some cases ? (*b*) Why is the benefit criterion more appropriate than the Regret or the Wald model ?

We first attempt to answer question (*a*) and on the basis of this answer, and the comparison of the 'benefit' criterion to those of Wald and Regret, we hope to answer (*b*).

The direct answer to (*a*) rests on the fact that there is a basic difference in definition of expected utilities for the three criteria. Due to the difference in definitions of $E(u_i)$, under Wald's criterion, the maximin principle is applied directly to P. Under the other two criteria, we convert P to R and B and then apply the maximin principle. Therefore, one should examine 'what happens when the original matrix P is converted to a regret or a benefit martix ?' This comparison is provided in figures 1 and 2 drawn for P, R and B in (1), (2) and (4) respectively. The two figures differ only in the sense that figure 1 has s_i on the x axis and t_j on the y axis whereas the reverse holds true for figure 2. The same nomenclature for two points in the two figures denotes that the two points are the same. For example, point J in figure 1 represents (s_1, t_3) or p_{13} with a value of 5. In figure 2 also, J represents p_{13}.

In figure 1 :

(*a*) t_1^P is the graph for the t_1 column of P, *i.e.* of the payoffs associated with s_1, s_2 and s_3 under t_1. This is DEF. Point D represents p_{11} [also called $p(s_1, t_1)$] which is the payoff to the decision maker when he employs his Ist strategy and the state of nature is t_1. Similarly, E represents p_{21} and F, p_{31}.

(*b*) t_2^P and t_3^P are the graphs for t_2 and t_3 colums respectively of P. These are GHI and JKL.

(*c*) t_1^R, t_2^R and t_3^R are drawn for R in (2). They are $D'E'F'$, $G'H'I'$ and $J'K'L'$ respectively.

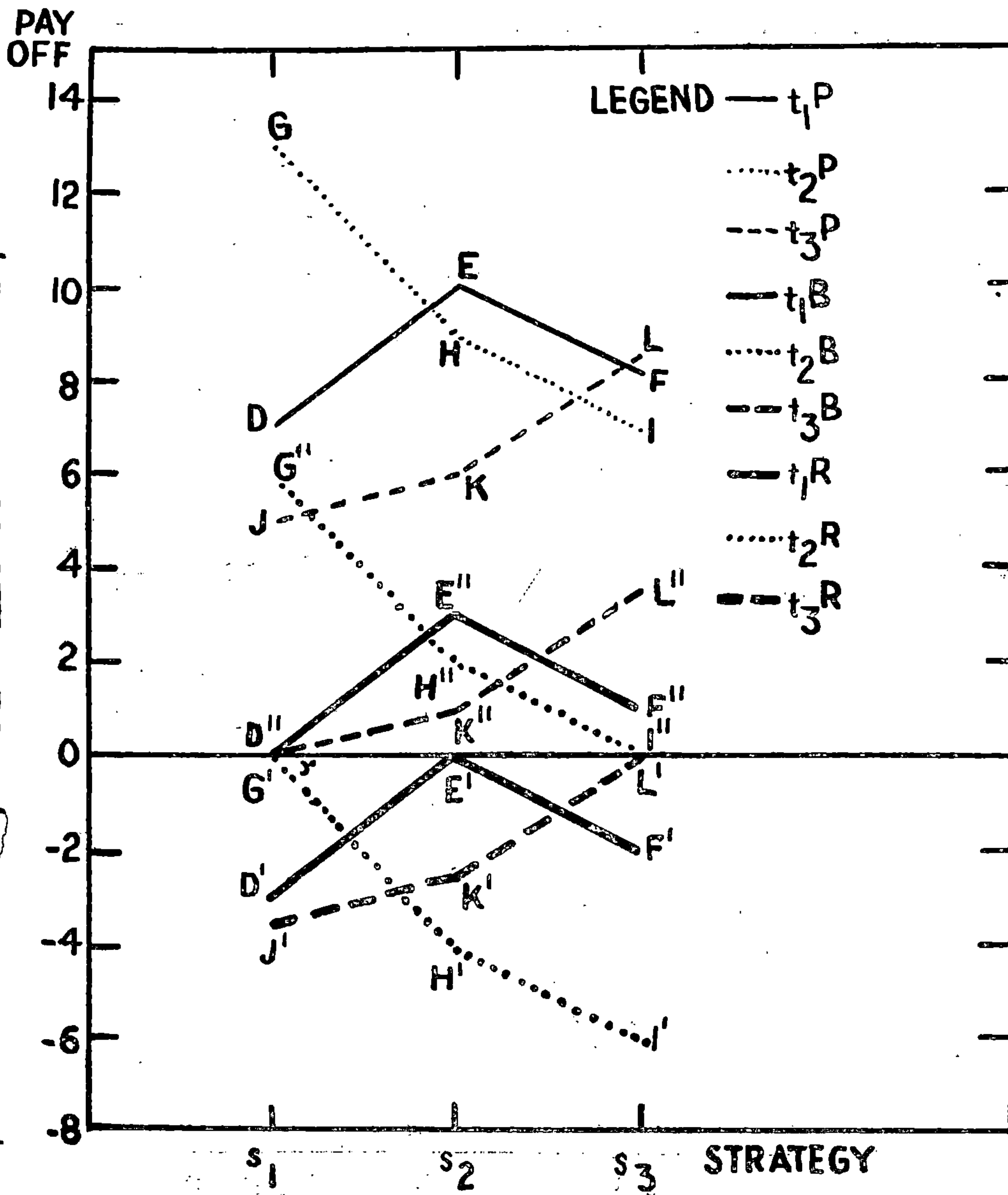

Figure 1. Payoffs for different strategies of nature

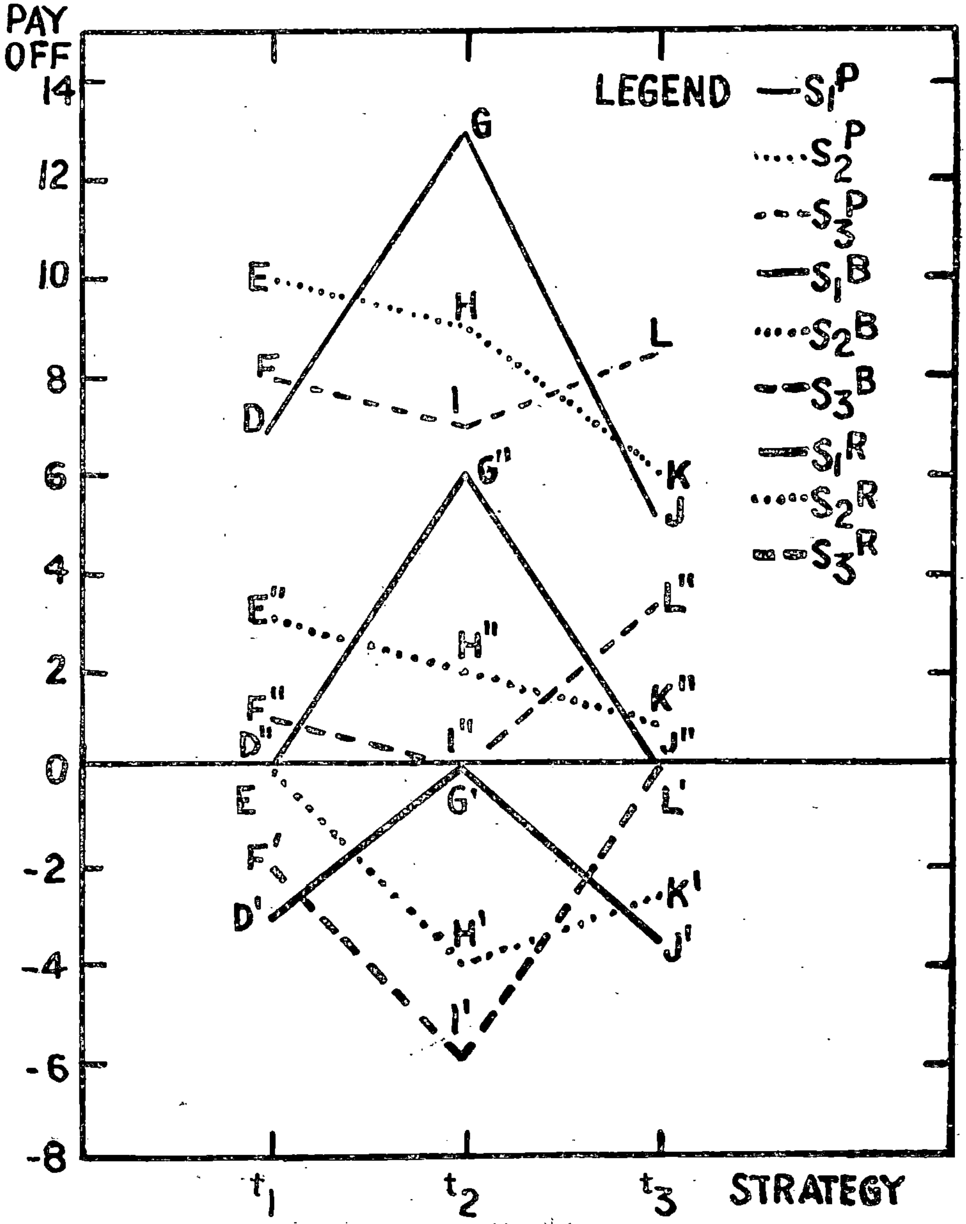

Figure 2. Payoffs for different strategies of player

(*d*) t_1^B ,t_2^B and t_3^B are labelled as $D''E''F''$, $G''H''I''$ and $J''K''L''$ and are for t_1, t_2 and t_3 columns of B.

Similarly, in figure 2 :

(*a*) s_1^P , s_2^P and s_3^P are the sets of the payoffs associates with the three rows of P.

(*b*) s_1^R , s_2^R and s_3^R are the graphs for R and s_1^B , s_2^B and s_3^B for B matrix.

In figure 1, t_1^P , t_1^R and t_1^B are parallel to each other ; the same holds true for t_2 and t_3. This property of t_{jo}^P , t_{jo}^R and t_{jo}^B being parallel to each other is indicative of the fact that when P is reduced to B or R, the position of s_i in relation to each other (in order of being highest, next, highest, ..., lowest) remains the same for a given t_{jo}. In deriving B from P, we deduct the minimum payoff in a given t_{jo} from other payoffs in the same column thus displacing t_{jo}^P to t_{jo}^B in the downward direction to the extent of the minimum payoff in P under t_{jo}. The $\underset{1}{\text{minimum}}\ p_{i1}=7$ and, therefore, t_1^P (DEF) is displaced by 7 units to t_1^B ($D''E''F''$) for each s_i. Similarly, t_2^P (GHI) and t_3^P (JKL) are shifted downwards to t_2^B ($G''H''I''$,) and t_3^B ($J''K''L''$) by 7 and 5 units respectively. In case of a 'regret' matrix, t_{jo}^P is displaced in the downward direction to t_{jo}^R and the displacement is equal to the $\underset{i}{\max_i}\ p_{ijo}$. For t_1, $\underset{i}{\max}\ p_{i1}=10$ and, therefore, t_1^P (DEF) is displaced to t_1^R ($D'E'F'$) by 10 units. Likewise, t_2^P (GHI) is shifted by 13 units to t_2^R ($G'H'I'$) and t_3^P (JKL) to t_3^R ($J'K'L'$) by 8 units. Obviously, the displacement of t_{jo} in case of R is always more than (or equal to) that in B because the maximum in a column of P is always greater than or equal to the maximum in that column. If $\underset{i}{\min_i}\ P_{ijo}=0$ for some j, then t_{jo}^P and t_{jo}^B are identical. Because the displacements of a given t_{jo}^P to t_{jo}^R and to t_{jo}^B are not identical, the three models will, in many instances, suggest different strategies as optimal.

(*a*) First let us consider s_1^P , s_2^P and s_3^P . s_1^P consists of points D, G and J such that G is the highest, D is next and J is the lowest, *i.e.* $G>D>J$. According to the definition, $E(u_1{}^W)=\min_j\ p_{1j}$. Because J is the lowest of the three points in s_1^P , J represents

$E(u_1^W)$. Similarly, in s_2^P, $E>H>K$ and, therefore, $E(u_2^W)$ is represented by K. $L>F>I$ in s_3^P and I represents $E(u_3^W)$. Further, $\max_i E(u_i^W)=E(u_{i*}^W)$. By looking at figure 2, we find that $I>K>J$ and, therefore, I represents $E(u_{i*}^W)$ which is obtained by using s_3.

(*b*) Now consider the graphs s_1^R, s_2^R and s_3^R for the regret matrix in (2). s_1^R consists of D', G' and J' which are displaced D, G and J of s_1^P. The order of these elements in s_1^R is $G'>D'>J'$. In s_2^R (E' H' K'), the order is $E'>K'>H'$ which differs from $E>H>K$ in s_2^P. The positions of H and K are interchanged. As pointed out earlier, this shift is due to the non-uniform displacement of t_j^P. The $E(u_1^R)$, $E(u_2^R)$ and $E(u_3^R)$ are represented by J', H' and I' respectively. From figures 1 and 2, we can see that J' H' I' and, therefore, J' represents $E(u_{i*}^R)$. Because J′ belongs to s_1^R, s_1 is the optimal strategy under the regret criterion.

(*c*) For s_1^B; s_2^B and s_3^B, $E(u_1^B)$ is represented by D'' or J'', $E(u_2^B)$ by H'' and $E(u_3^B)$ by I″. Because H″ is the highest of these points, $\max_i E(u_i^B)=E(u_2^B)$, and s_2 is optimal under benefit criterion.

Thus, we see why the three criteria may suggest different strategies as optimal for a given P under some situations. However, this may not always be true. One situation, for example, would be where one strategy of the decision maker (row of P) *strictly* dominates his all other strategies. In this case, no matter what criterion we apply, the *strictly* dominating strategy would always come out to be the best.

Characteristics of the 'Benefit' Model relative to the Wald & Regret Models :

Benefit and Regret matrices (B & R) are similar in the sense that both are derived from P by deducting a constant from other payoffs for a given state of nature, Since, in deriving R from P, the maximum in a column of P is considered, the approach is oather optimistic. In calculating the 'benefit' matrix, the worst consequence for a given state of nature is taken into account. Therefore, this approach is more conservative than the 'Regret' approach. If min

$p_{ij}=0$ for all j, then $P=B$ and $E(u_i^W)=E(u_i^B)$ for all i. In this case, the Wald and the 'Benefit' models will give identical results. Due to the property of column-linearity, the principle of 'insufficient reason' suggests the same strategy no matter on which matrix (P, R or B) it is applied. The 'Benefit' approach combines the Wald and Regret models as is brought out in figure 2. Let us look at s_2^P, s_2^R and s_2^B. While s_2^P ($E\ H\ K$) has a kind on its upper side at H, the regret criterion curve s_2^R ($E'\ H'\ K'$), has a kink on its lower side at H'; and a graph of s_2 for the 'benefit' criterion, s_2^B, is a straight line $E''\ H''\ K''$ representing a compromise between the Wald (rank pessimistic) and Savage's Regret (optimistic) criteria. Therefore, the benefit approach can be termed as 'neither too optimistic nor too conservative'. Under the most unfavourable states of nature, the 'benefit' approach may not always be as good as the Wald model. Under extremely favourable conditions, it may be inferior to the 'regret' principle. However, under the greater number of 'in between' cases, it appears to be superior, more realistic and, therefore, more practical than any other criterion. One of its advantages is that, for a given state of nature, it allows the decision maker a payoff higher than the worst in most cases. Usually, then it protects the decision maker from the worst consequences for a given state of nature, t_j. It needs to be examined under various numerical situations (payoff matrices). However, as a hybrid between the Wald and Savage's Regret approaches, the nature of many payoff matrices may suggest the utility of the 'benefit' criterion.

REFERENCES

1. Agarwal, R.C. Applications of operations research techniques in agriculture. Unpublished Ph.D. thesis, Ames, Iowa, Library, Iowa State University of Science and Technology, 1967.
2. Agarwal, R.C. and Heady, E.O. Applications of game theory models in agriculture. Journal of Agricultural Economics 19 : *207* : 207-218. 1968.
3. Hurwicz, L. Optimality criteria for decision making under ignorance. Mimeo. Cowles Commission Discussion Paper 350. 1951.
4. Luce, R.D. and Raiffa, H. Games and decisions. New York, New York, John Wiley & Sons. 1957.
5. Savage, L.J. The theory of statistical decisions. American Statistical Association Journal *46* : 55-67. 1951.
6. von Neumann, J. and Morgenstern, O. Theory of games and economic behavior. 3rd ed. Princeton, New Jersey, Princeton University Press, 1953.
7. Wald, A. Statistical decision functions. New York, New York, John Wiley and Sons. 1950.